

# Monte Carlo Valuation of Future Annuity Values

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1. Introduction
2. Problem statement and proposed methodology
  - Annuity valuation with the Least-Squares Monte Carlo Approach
3. A numerical example
4. Validation
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## INTRODUCTION

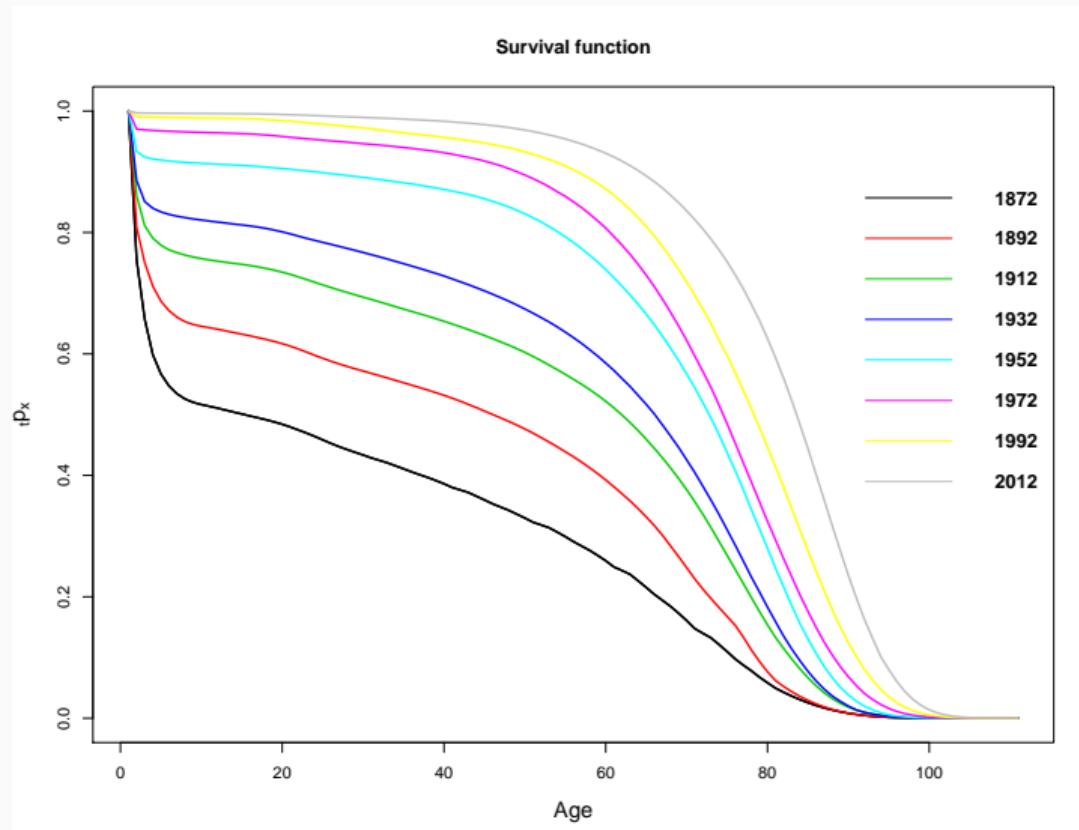
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- Due to health improvements and medical advances, people tend to live longer and longer.

Longevity risk: risk to which pension funds/life insurance companies are exposed to as a result of ever-increasing life expectancy trends among policyholders/pensioners (Blake et al., 2013, Pitacco, 2002, Willets et al., 2004, Wilmoth, 2000).

- Can result in higher payout levels;
- Can lead to possible solvency issues;
- Need models to reliably anticipate future mortality paths.

# SURVIVAL FUNCTION: ITALY 1872-2012 (MALE)



Future annuity values are **uncertain**:

- Unknown future mortality and interest rates;
- Impact on liabilities for insurers/pension plans ([Oppers et al., 2012](#));
- Impact on dependence between lifetimes ([Alai et al., 2013, 2015](#) and [Alai, 2019](#));
- By product: distribution of future life expectancy.

## PROBLEM STATEMENT AND PROPOSED METHODOLOGY

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Propose a simulation based method to evaluate the distribution of future annuity values.

- Avoid nested simulations;
- Use the well-known LSMC method;
- Flexible to accommodate any (Markov) mortality model;
- Extend to more general situations (see later).

Life annuity: contract providing a series of payments at fixed time intervals, while the purchaser (or annuitant) is alive.

- Value of an immediate annuity prevailing at time  $T > 0$  for an individual aged  $x + T$  at time  $T$  is

$$a_{x+T}(T) = \sum_{i=1}^{+\infty} B(T, T+i) {}_i p_{x+T}(T) \quad (1)$$

where

- $B(T, T+i)$ :  $i$ -th years discount factor prevailing at time  $T > 0$ ;
- ${}_i p_y(T)$ :  $i$ -th years survival probability for an individual aged  $y$  at time  $T$ .

- $B(T, T + i)$ ,  ${}_i p_{x+T}(T)$ , and  $a_{x+T}(T)$ : random variables at time 0 (today).
  - These random variables are **conditional expectations** given the information available at time  $T$ .
- What do we need?
  - Both  $B(T, T + i)$  and  ${}_i p_{x+T}(T)$  require models for the (stochastic) evolution of interest rates and mortality rates.
- Warning
  - Closed form formulae for the corresponding conditional expectations are not guaranteed!

To estimate (1), a straightforward method would rely on a simulation within simulation framework (nested simulations).

## Algorithm

1. Simulate all relevant financial and demographic factors up to time  $T$ ;
2. For each simulated time  $T$  value of relevant factors, simulate forward starting from that particular value;
3. Evaluate the conditional expectation by averaging.

It is a very time consuming procedure!

## Task

- Reducing the complexity of computations preserving, at the same time, the accuracy of the estimates.

## State of the art

- Cairns (2011), Dowd et al. (2011) and Liu (2013) propose an approach based on a Taylor series approximation of the conditional expectation but still requiring multiple simulation sets;
- Denuit (2008) proposes comonotonic approximations to quantiles of life annuity conditional expected present value.

## PROBLEM STATEMENT AND PROPOSED METHODOLOGY

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**ANNUITY VALUATION WITH THE  
LEAST-SQUARES MONTE CARLO  
APPROACH**

Provide an alternative methodology for determining future annuity values.

- Exploit the well-established Least-Squares Monte Carlo (LSMC) approach.

## The idea

Using Monte Carlo method combined with regression to approximate the conditional expectations and calculate the annuity value.

## Advantage

- Flexibility, as it is implementable regardless of model complexity.

## Assumptions

- Deterministic interest rate level;
- Stochastic evolution of mortality;
- Unitary amount of benefits paid at regular intervals.

## Algorithm

1. Simulate future mortality patterns;
2. Use regression across different simulations (LSMC) to approximate annuity values.

- Consider (as an example) the Poisson version of the Lee-Carter model;

### Model specification

- Number of deaths at age  $x$ , year  $t$ ,

$$D_{x;t} \sim \text{Poisson}(E_{x;t} m_{x;t}),$$

where  $E_{x;t}$  is central exposure;

- The central death rate at age  $x$  in year  $t$ ,  $m_{x;t}$ :

$$\log m_{x;t} = \alpha_x + \beta_x \kappa_t,$$

where

- $\alpha_x$  and  $\beta_x$  are two age-specific parameters;
- $\kappa_t$  is a time-index parameter (random walk with drift).

## Conditional survival probabilities

Since  $\kappa_t$  is modelled as a Markov process:

$${}_i p_{x+T} = \mathbb{E} \left[ e^{-(m_{x+T; T+\dots+m_{x+T+i-1; T+i-1}})} \mid \kappa_T \right].$$

## Expected value of annuity payments

$$a_{x+T}(T) = \mathbb{E} \left[ \sum_{i=1}^{\omega-T-x} e^{-ir} e^{-(m_{x+T; T+\dots+m_{x+T+i-1; T+i-1}})} \mid \kappa_T \right] \quad (2)$$

where  $\omega$  is the ultimate age and  $r$  the discount rate.

- Fit Poisson Lee-Carter on mortality data from HMD:
  - Italian male population 1965 – 2014;
  - Ages 0 – 90;
  - Number of simulations  $N = 10000$ .
- Financial factor:
  - Discount rate  $r = 0.03$ .

## Steps

1. Simulate  $\kappa_t^{(j)}$  and  $m_{x;t}^{(j)}$ ,  $t = 2015, 2016, \dots$ ,  $j = 1, \dots, N$ ;
2. Compute

$$A^{(j)} = \sum_{i=1}^{\omega-x-T} e^{-ir} e^{-\left(m_{x+T;T}^{(j)} + \dots + m_{x+T+i-1;T+i-1}^{(j)}\right)}, \quad j = 1, \dots, N.$$

### 3. Regress

$$\left\{ A^{(j)} \right\}_j \text{ on } \left\{ b \left( \kappa_T^{(j)} \right) \right\}_j$$

where  $b = (b_1, \dots, b_p)$ , e.g. powers:

$$A^{(j)} = \beta_0 + \beta_1 \kappa_T^{(j)} + \dots + \beta_p \left( \kappa_T^{(j)} \right)^p + \epsilon^{(j)};$$

4. Estimate  $\hat{\beta}_0, \dots, \hat{\beta}_p$  via OLS;
5. Compute

$$\hat{a}_{x+T}^{(j)}(T) = \hat{\beta}_0 + \hat{\beta}_1 \kappa_T^{(j)} + \dots + \hat{\beta}_p \left( \kappa_T^{(j)} \right)^p \quad j = 1, \dots, N.$$

## A NUMERICAL EXAMPLE

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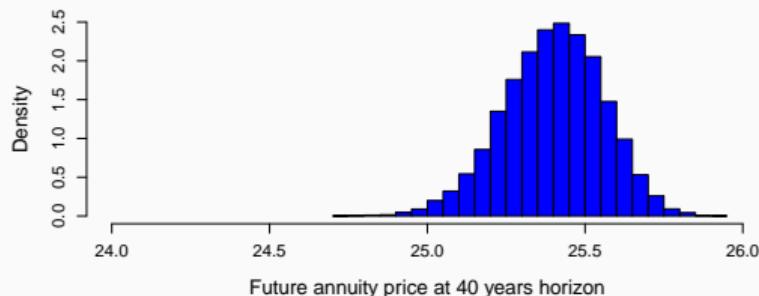
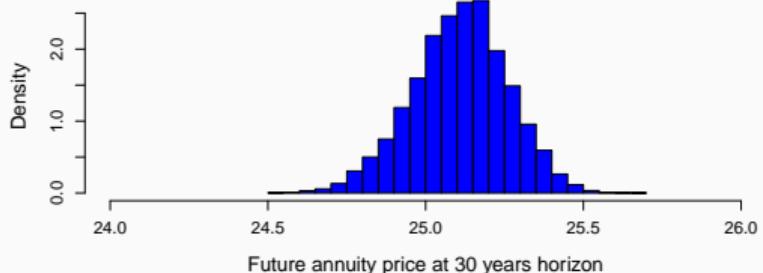
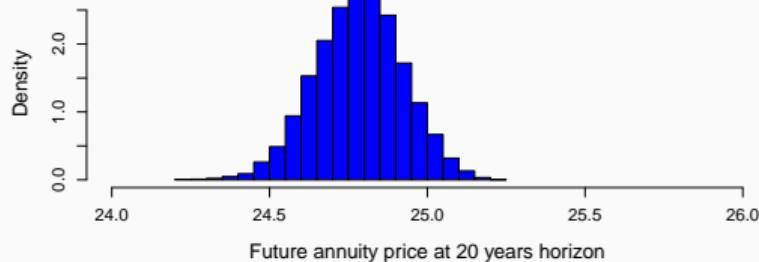
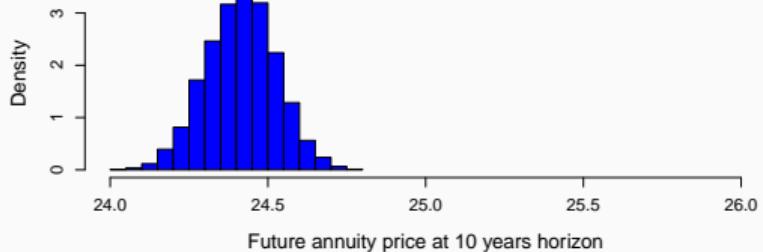
Distribution of annuity values at time horizon  $T$  for individuals aged 40 in year  
 $2014 + T$

$$a_{40}(0) = 23.72$$

	Mean	Std Dev	Skewness	Kurtosis	10th perc.	90th perc.
<b>T=10</b>	24.42	0.11	-0.06	2.91	24.27	24.56
<b>T=20</b>	24.78	0.14	-0.11	3.02	24.61	24.96
<b>T=30</b>	25.11	0.15	-0.19	3.04	24.91	25.30
<b>T=40</b>	25.40	0.16	-0.24	3.05	25.20	25.60

# ANNUITY VALUE DISTRIBUTION |

Aged 40



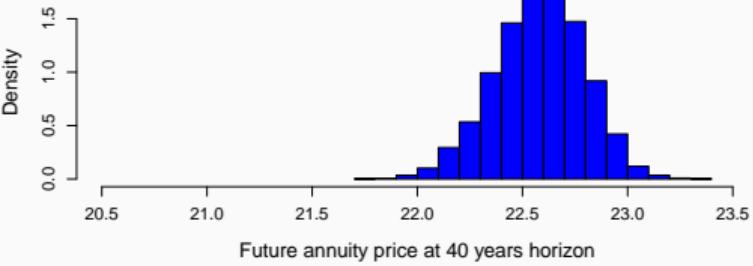
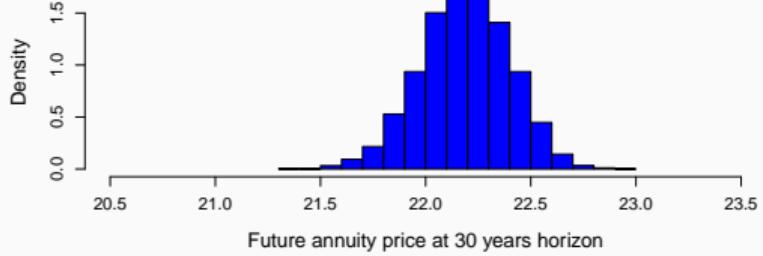
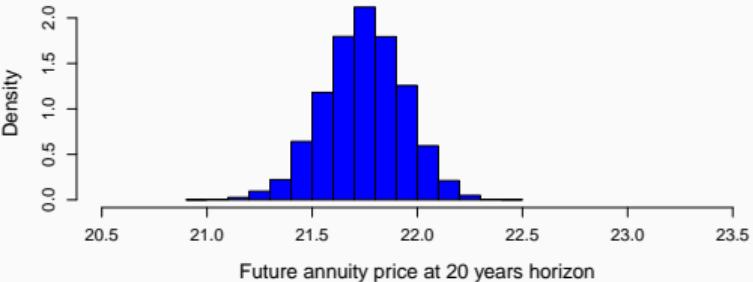
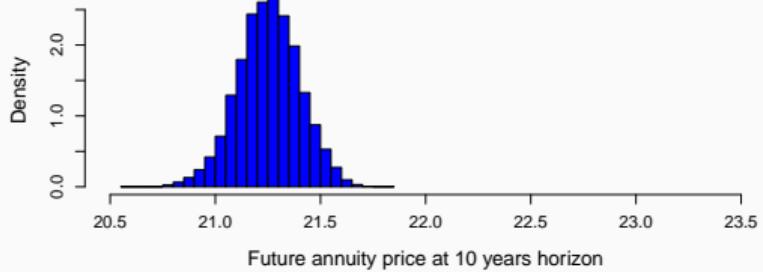
Distribution of annuity values at time horizon  $T$  for individuals aged 50 in year  
 $2014 + T$

$$a_{50}(0) = 20.32$$

	Mean	Std Dev	Skewness	Kurtosis	10th perc.	90th perc.
<b>T=10</b>	21.25	0.15	-0.10	3.12	21.07	21.44
<b>T=20</b>	21.74	0.19	-0.13	3.03	21.50	21.99
<b>T=30</b>	22.19	0.21	-0.13	2.99	21.92	22.46
<b>T=40</b>	22.58	0.21	-0.18	2.97	22.30	22.85

# ANNUITY VALUE DISTRIBUTION II

Aged 50



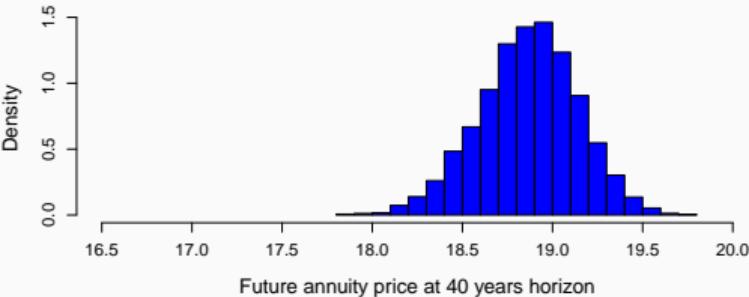
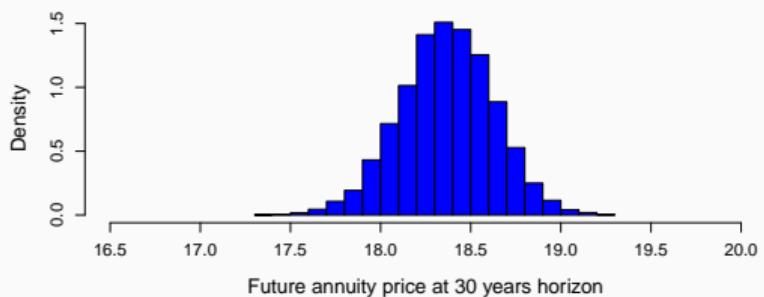
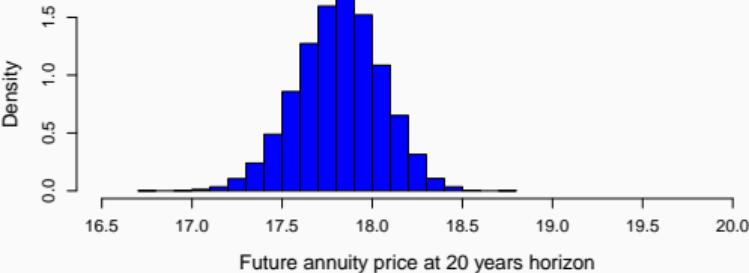
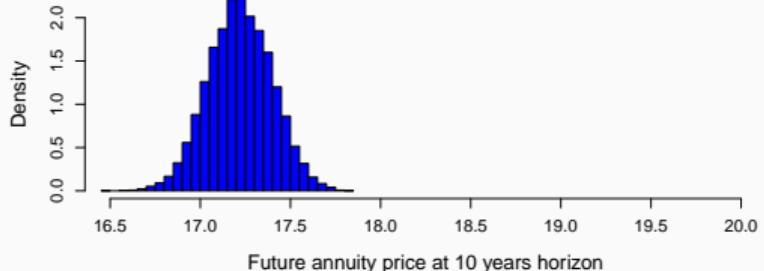
Distribution of annuity values at time horizon  $T$  for individuals aged 60 in year  
 $2014 + T$

$$a_{60}(0) = 16.09$$

	Mean	Std Dev	Skewness	Kurtosis	10th perc.	90th perc.
<b>T=10</b>	17.22	0.18	-0.04	2.93	16.99	17.45
<b>T=20</b>	17.82	0.23	-0.14	2.97	17.52	18.11
<b>T=30</b>	18.37	0.26	-0.09	3.03	18.03	18.69
<b>T=40</b>	18.87	0.27	-0.19	3.04	18.50	19.21

# ANNUITY VALUE DISTRIBUTION III

Aged 60



For fixed age  $x$ , the distribution of annuity values changes as the future time  $T$  increases:

- Mean increases;
- Standard Deviation increases;
- The distribution becomes increasingly left-skewed;
- Kurtosis slightly increases.

These effects are amplified as the age increases.

## VALIDATION

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A solid benchmark can be obtained through a nested simulation procedure.

## Steps

1. Simulate  $\kappa_t^{(j)}$  and  $m_{x;t}^{(j)}$ ,  $t = 2015, 2016, \dots, (2015 + T)$   $j = 1, \dots, N$ ;
2. Starting from  $\kappa_T^{(j)}$  and  $m_{x;T}^{(j)}$ , we simulate  $S$  inner scenarios

$$\kappa_t^{(j;s)} \text{ and } m_{x;t}^{(j;s)},$$

where  $t = (2016 + T), (2017 + T), \dots, j = 1, \dots, N$  and  $s = 1, \dots, S$ ;

3. Compute

$$A^{(j;s)} = \sum_{i=1}^{\omega-x-T} e^{-ir} e^{-\left(m_{x+T;T}^{(j;s)} + \dots + m_{x+T+i-1;T+i-1}^{(j;s)}\right)}, \quad j = 1, \dots, N \text{ and } s = 1, \dots, S.$$

## Least-Squares Monte Carlo Algorithm

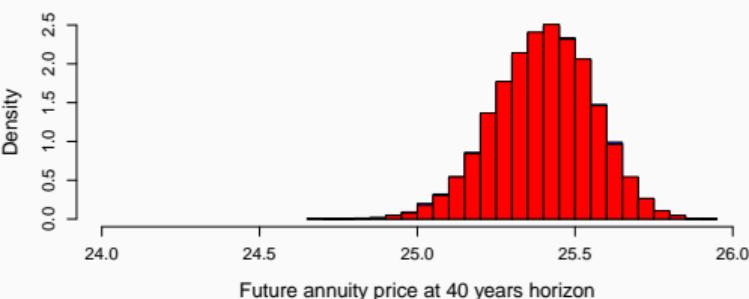
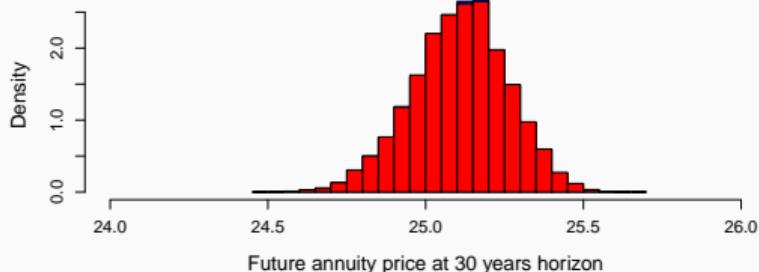
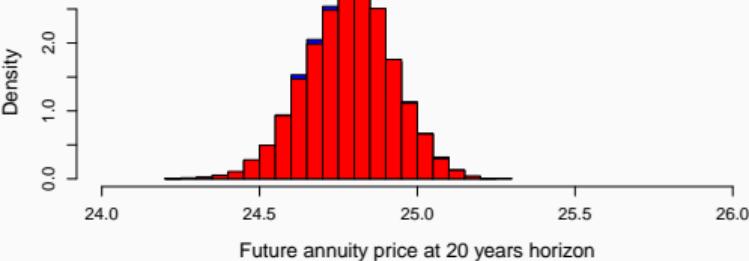
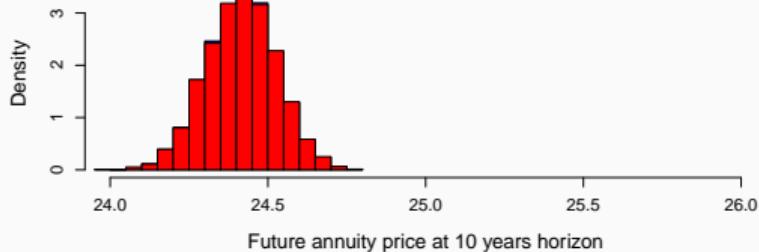
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## Nested Simulation Algorithm

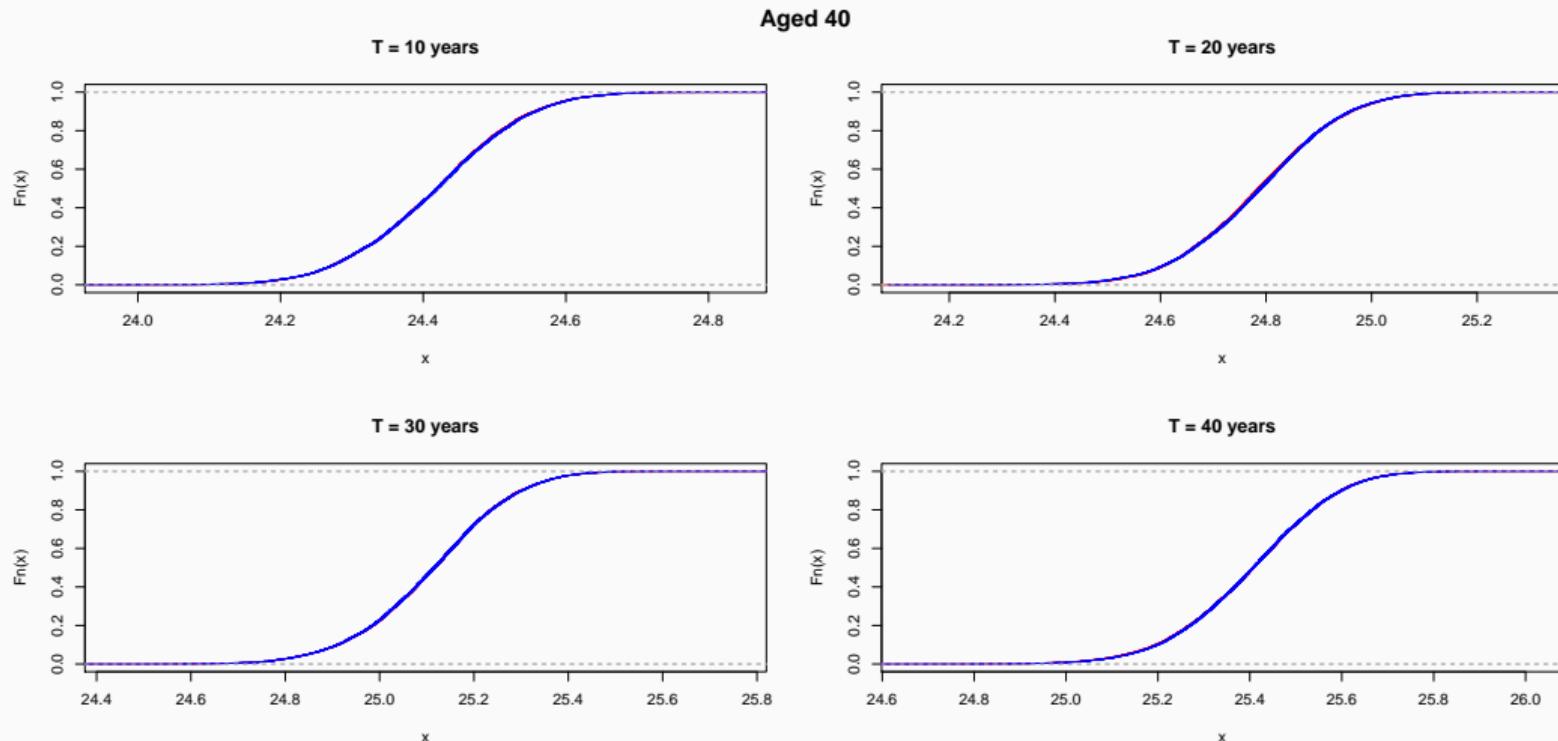
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# RESULTS: ANNUITY VALUE DISTRIBUTION

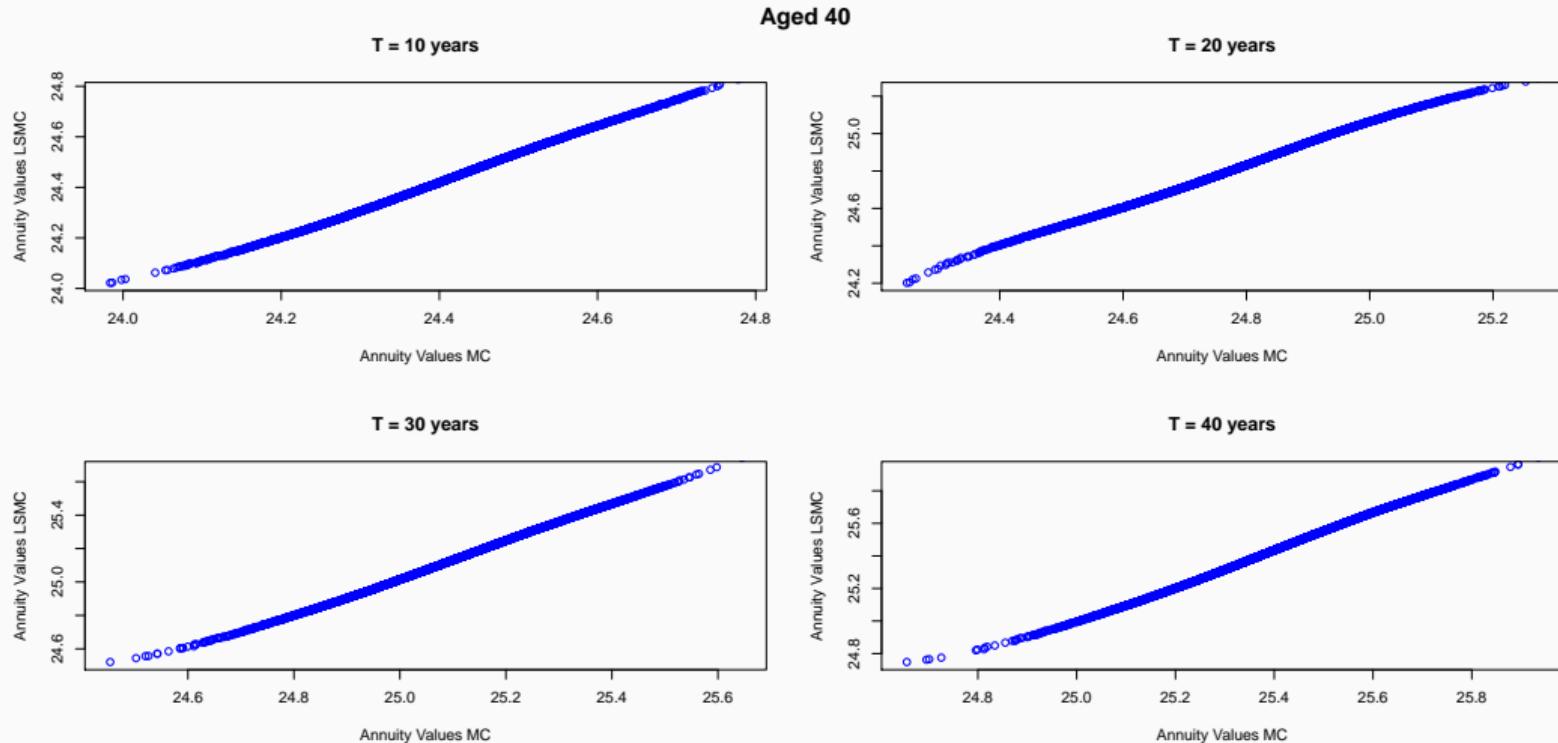
Aged 40



# RESULTS: EMPIRICAL DISTRIBUTION FUNCTION



# RESULTS: Q-Q PLOT



## Least-Squares Monte Carlo Algorithm

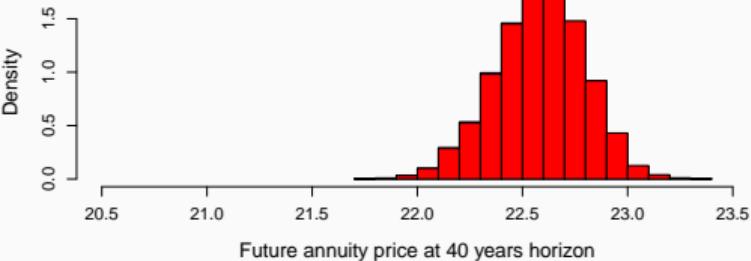
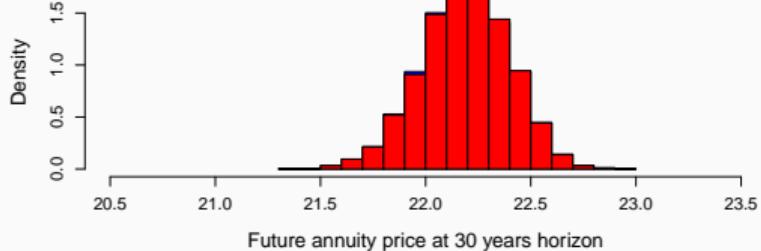
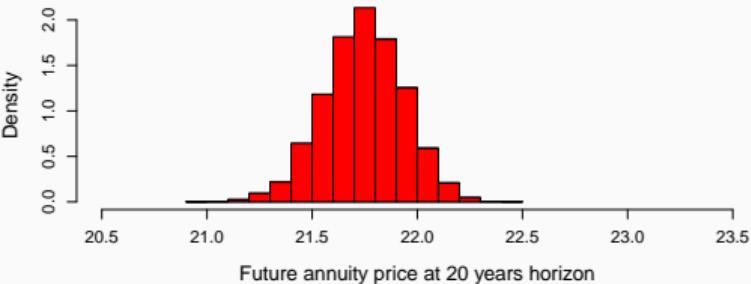
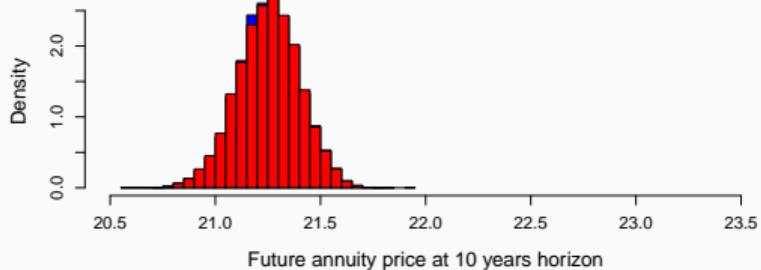
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## Nested Simulation Algorithm

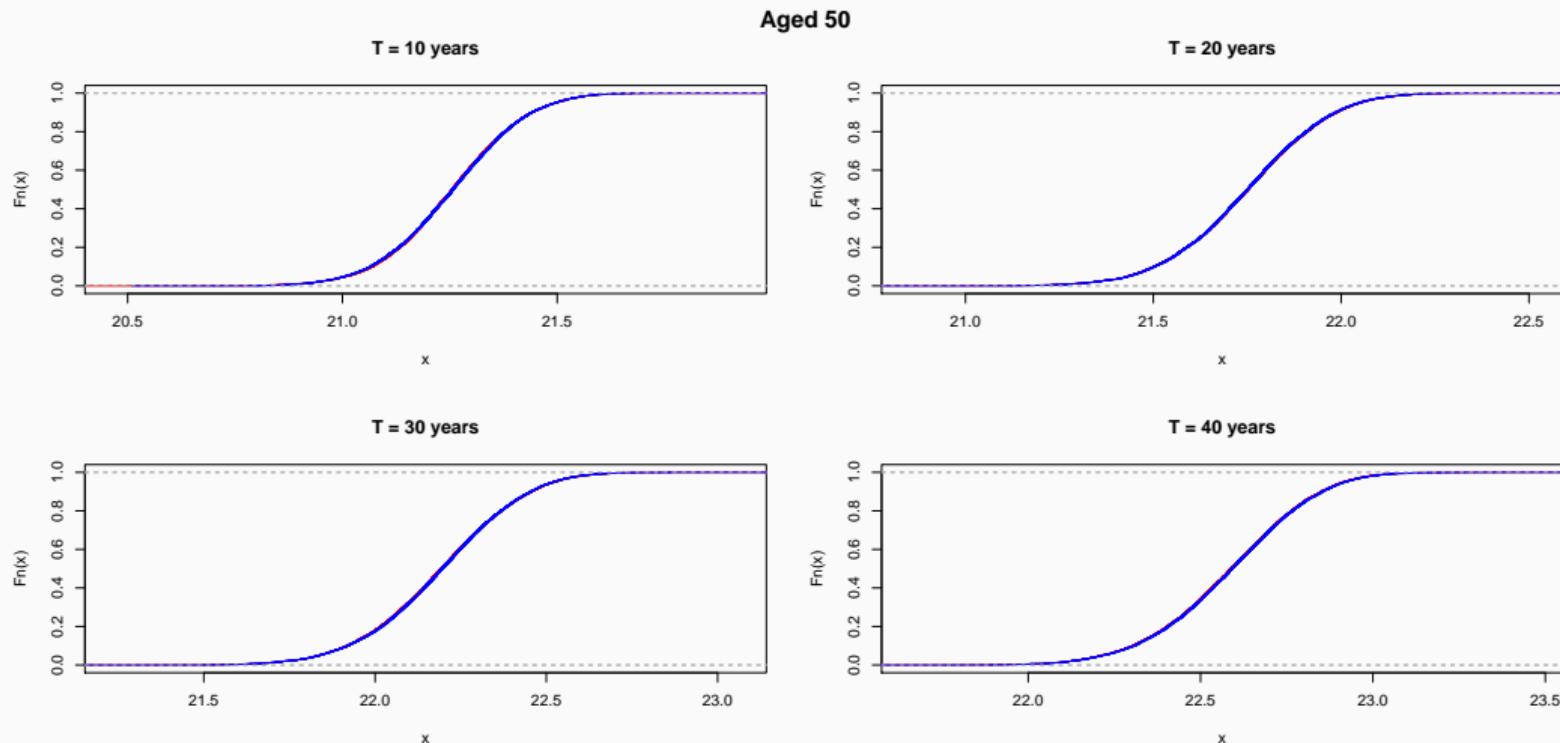
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# RESULTS: ANNUITY VALUE DISTRIBUTION

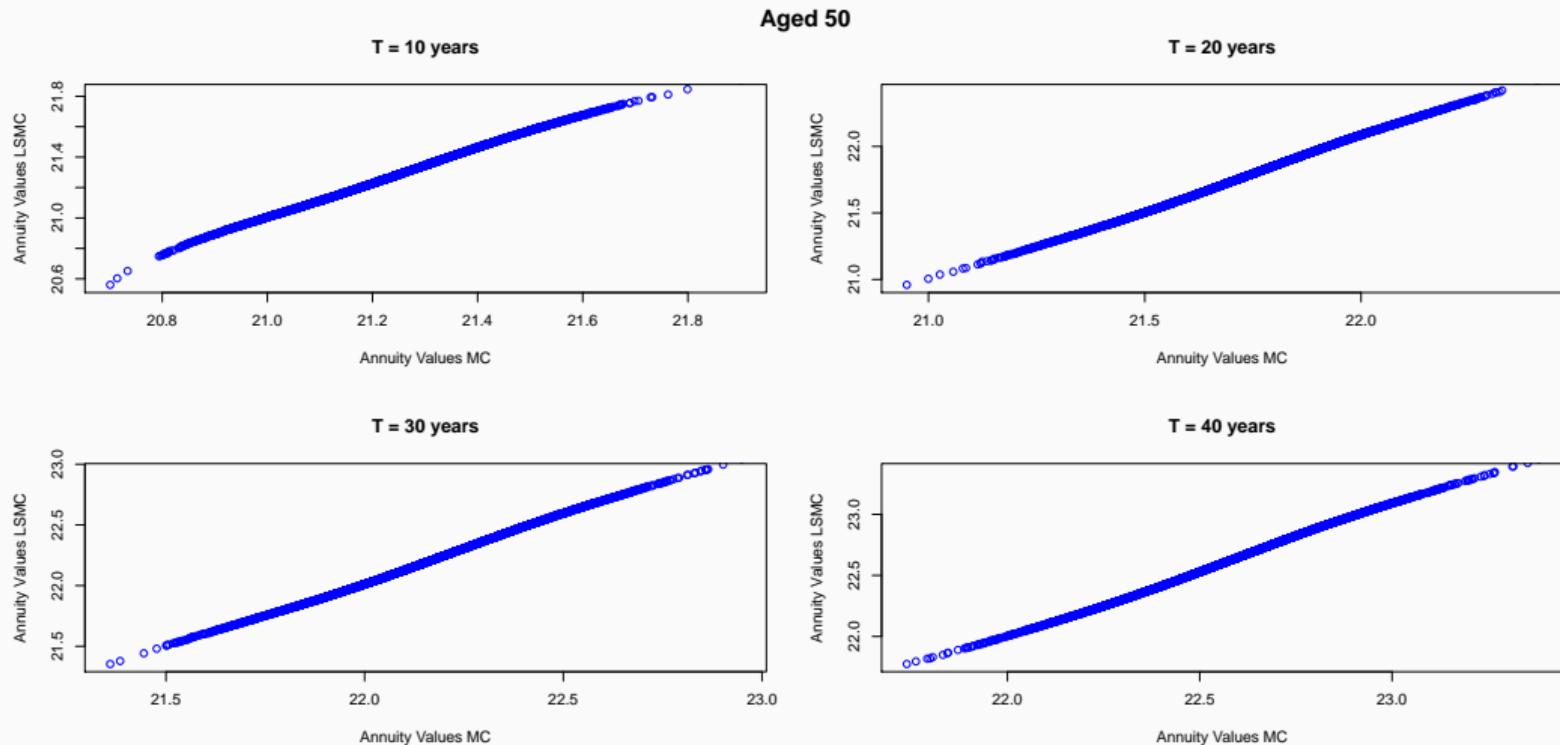
Aged 50



# RESULTS: EMPIRICAL DISTRIBUTION FUNCTION



# RESULTS: Q-Q PLOT



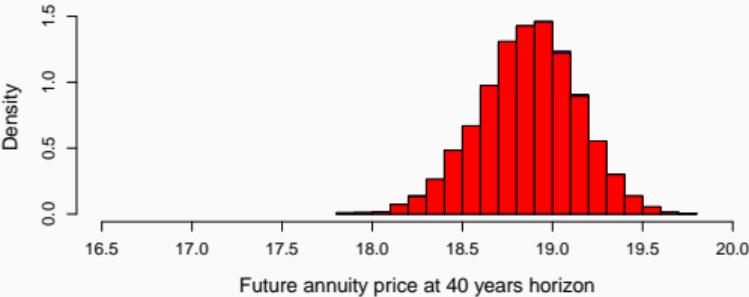
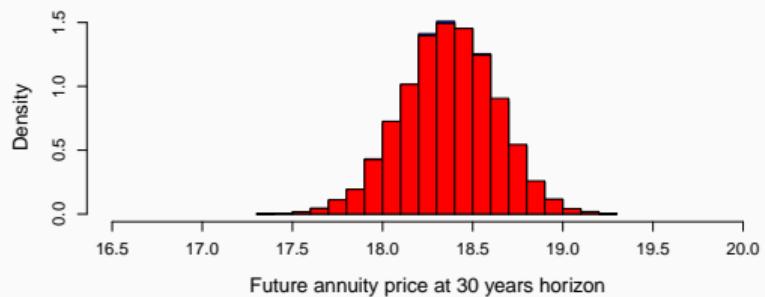
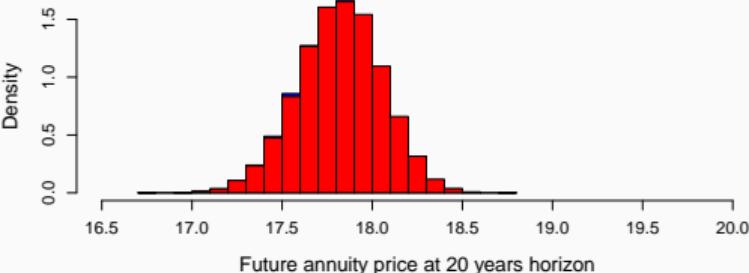
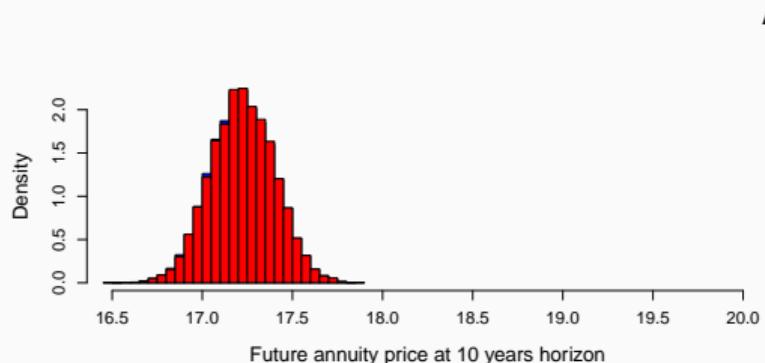
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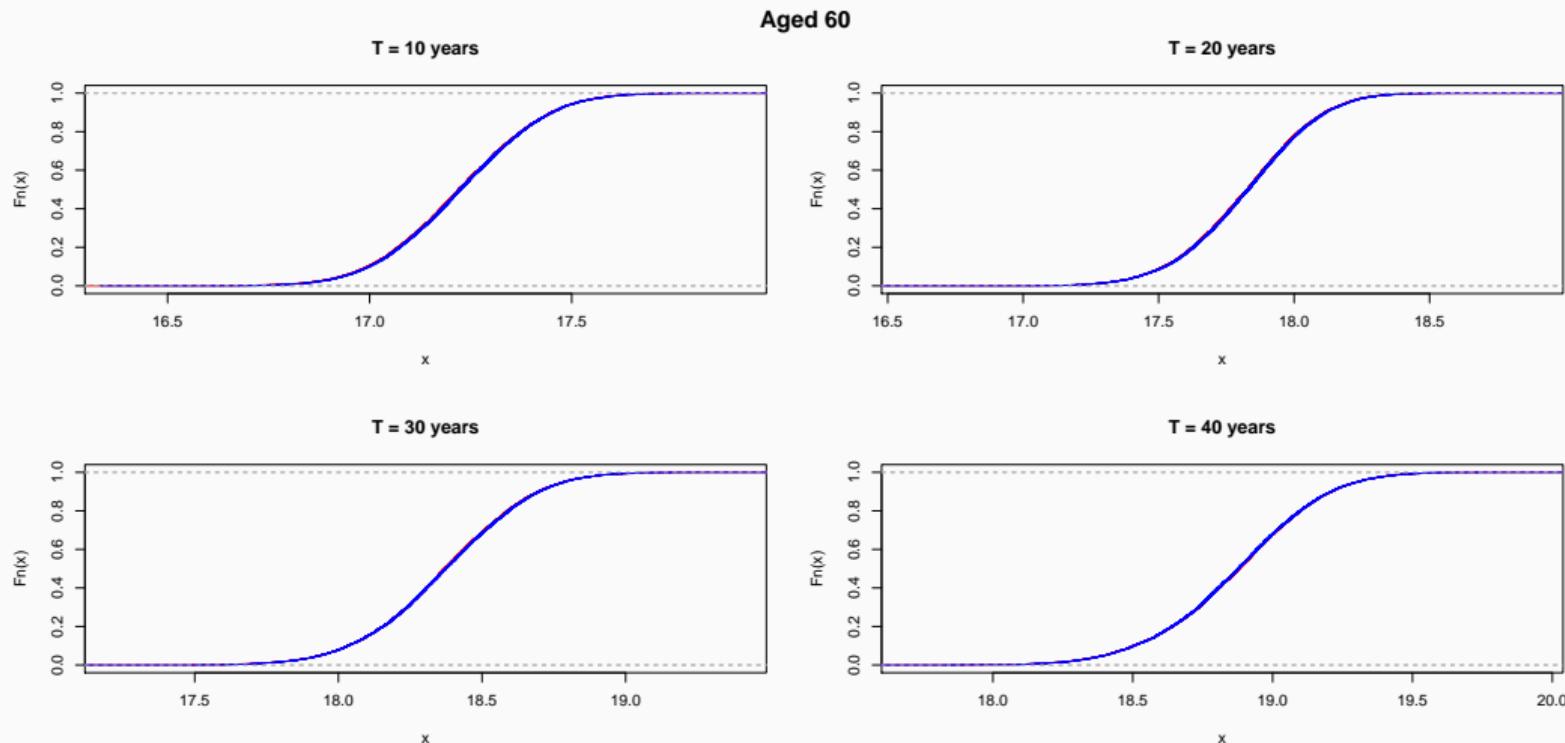
## Nested Simulation Algorithm

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<b>T=20</b>	17.82	0.13	-0.13	3.01	17.52	18.12
<b>T=30</b>	18.37	0.26	-0.09	2.97	18.03	18.70
<b>T=40</b>	18.87	0.27	-0.18	3.06	18.50	19.21

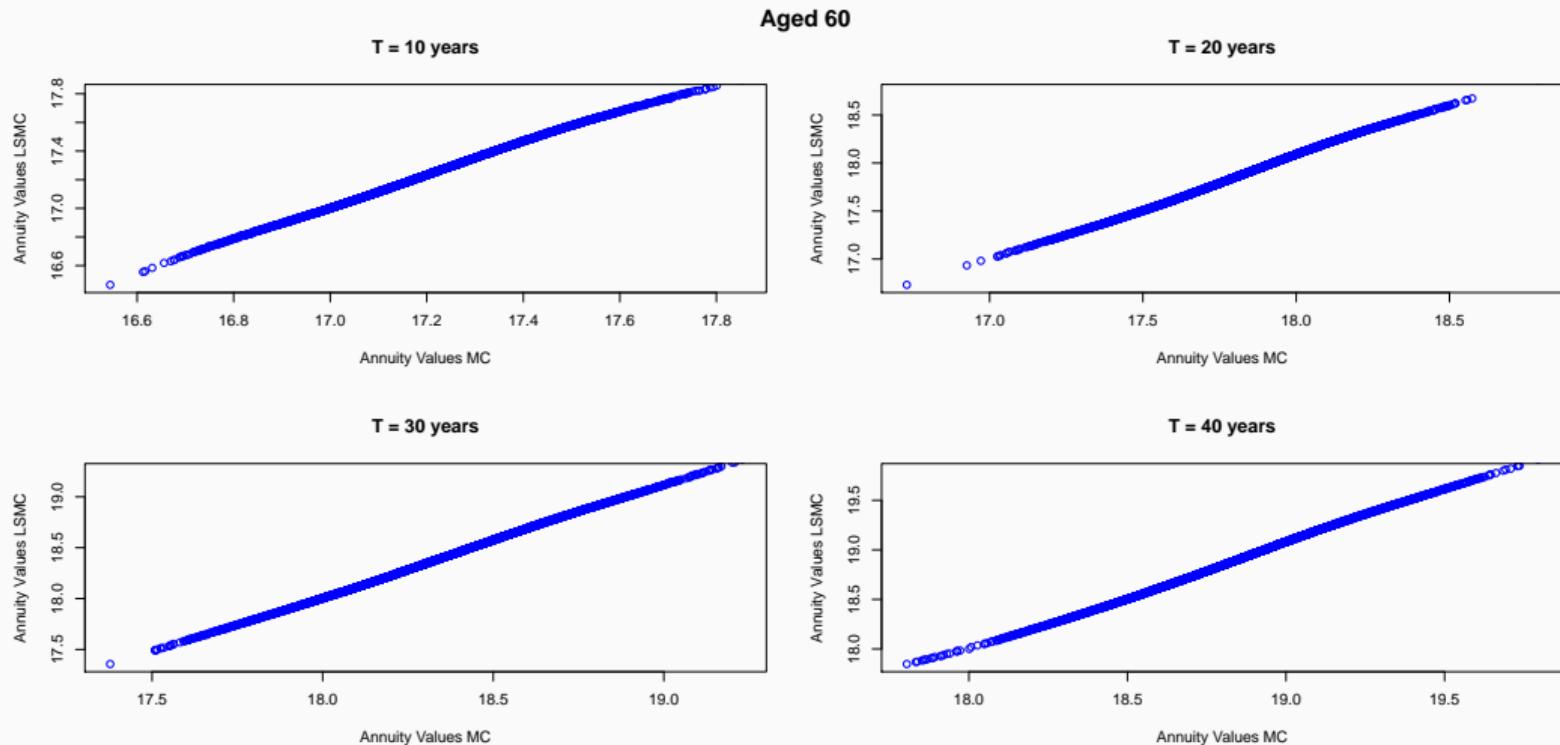
# RESULTS: ANNUITY VALUE DISTRIBUTION



# RESULTS: EMPIRICAL DISTRIBUTION FUNCTION



# RESULTS: Q-Q PLOT



How many times the LSMC estimates fall in the 95% confidence interval?

	Age 40			Age 50			Age 60		
	Left	Inside	Right	Left	Inside	Right	Left	Inside	Right
<b>T=10</b>	8.09%	87.68%	4.23%	7.13%	84.46%	8.41%	22.14%	76.77%	1.09%
<b>T=20</b>	12.57%	80.43%	7.00%	4.79%	85.99%	9.22%	24.10%	75.04%	0.86%
<b>T=30</b>	5.35%	87.50%	7.15%	14.93%	79.47%	5.60%	17.07%	80.73%	2.20%
<b>T=40</b>	12.67%	83.47%	3.86%	21.60%	77.15%	1.25%	6.24%	84.79%	8.97%

### Kolmogorov-Smirnov Test

	KS Stat. Age 40	KS Stat. Age 50	KS Stat. Age 60
<b>T=10</b>	0.0059 (0.9950)	0.0071 (0.9626)	0.0086 (0.8534)
<b>T=20</b>	0.0094 (0.7689)	0.0051 (0.9995)	0.0070 (0.9671)
<b>T=30</b>	0.0036 (0.9997)	0.0068 (0.9749)	0.0059 (0.9950)
<b>T=40</b>	0.0053 (0.9990)	0.0055 (0.9982)	0.0055 (0.9998)

## FUTURE EXTENSIONS

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- Stochastic interest rate;
- More complicate mortality models;
- Other types of annuities (e.g. variable annuities, equity indexed, . . . ):
  - Pricing and risk-management purposes;
- Evaluate pension plans Buy-ins and -outs;
- Comparison with Dowd et al. (2011);
- Explore variants of algorithm;
- R code.

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**Thank you!**